Chapter 6: Lecture 25 Linear Algebra, Course 124B, Fall, 2008

Prof. Peter Dodds

Department of Mathematics & Statistics University of Vermont



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All the way with $A\vec{x} = \vec{b}$:

- Applies to any $m \times n$ matrix A.
- ► Symmetry of *A* and *A*^T.

Where \vec{x} lives:

- ▶ Row space $C(A^{\mathrm{T}}) \subset R^{n}$.
- ▶ (Right) Nullspace $N(A) \subset R^n$.
- dim $C(A^{T})$ + dim N(A) = r + (n r) = n
- Orthogonality: $C(A^{\mathrm{T}}) \bigotimes N(A) = R^n$

Where \vec{b} lives:

- Column space $C(A) \subset R^m$.
- Left Nullspace $N(A^{\mathrm{T}}) \subset R^{m}$.
- dim C(A) + dim $N(A^{T}) = r + (m r) = m$
- Orthogonality: $C(A) \bigotimes N(A^{T}) = R^{m}$

Outline

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Approximating

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The fundamental

theorem of linear

Approximating

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algebra

algebra

The fundamental theorem of linear algebra

Approximating matrices with SVD The basic idea Guess who? Bonus example 1

Bonus example 2

Best solution \vec{x}_* when $\vec{b} = \vec{p} + \vec{e}$:

Null Space

d = n

Row Space d = r $\vec{x_r} = \vec{p}$ $\vec{x_r} = \vec{p}$ $\vec{x_r} = \vec{p}$ $\vec{x_r} = \vec{x_r} + \vec{x_n}$ $\vec{x_n} = \vec{0}$ $\vec{x_n} = \vec{0}$ Column Space \vec{p} \vec{p} d = r $\vec{d} = r$ \vec{c} Left Null Space The fundamental theorem of linear algebra Approximating

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The fundamental theorem of linear algebra

 \mathbf{R}^m

d = m - r

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Fundamental Theorem of Linear Algebra

Now we see:

- Each of the four fundamental subspaces has a 'best' orthonormal basis
- ► The \hat{v}_i span R^n
- We find the \hat{v}_i as eigenvectors of $A^{\mathrm{T}}A$.
- ► The \hat{u}_i span R^m
- We find the \hat{u}_i as eigenvectors of AA^{T} .

Happy bases

- ${\hat{v}_1, \ldots, \hat{v}_r}$ span Row space
- ${\hat{v}_{r+1}, \dots, \hat{v}_n}$ span Null space
- $\{\hat{u}_1, \ldots, \hat{u}_r\}$ span Column space
- $\{\hat{u}_{r+1}, \ldots, \hat{u}_m\}$ span Left Null space

Image approximation (80x60)

Idea: use SVD to approximate images

- Interpret elements of matrix A as color values of an image.
- Truncate series SVD representation of A:

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\mathrm{T}} = \sum_{i=1}^{r} \sigma_{i}\hat{\boldsymbol{u}}_{i}\hat{\boldsymbol{v}}_{i}^{\mathrm{T}}$$

- Use fact that $\sigma_1 > \sigma_2 > \ldots > \sigma_r > 0$.
- ▶ Rank $r = \min(m, n)$.
- Rank r = # of pixels on shortest side.
- ► For color: approximate 3 matrices (RGB).

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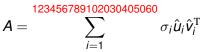
algebra

Fundamental Theorem of Linear Algebra

How $A\vec{x}$ works:

- $\blacktriangleright A = U\Sigma V^{\mathrm{T}}$
- ► A sends each $\vec{v}_i \in C(A^T)$ to its partner $\vec{u}_i \in C(A)$ with a stretch/shrink factor $\sigma_i > 0$.
- A is diagonal with respect to these bases and has positive entries (all σ_i > 0).
- When viewed the right way, any A is a diagonal matrix Σ.

Image approximation (80x60)



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The fundamental theorem of linear algebra

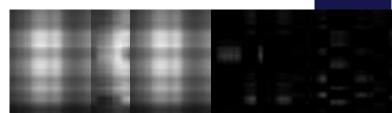
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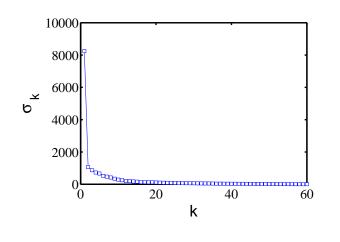
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Decay of sigma values: Einstein





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Image approximation (480x615)

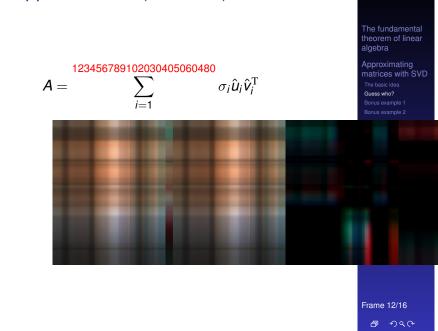
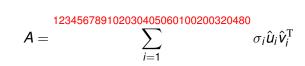


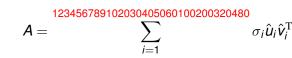
Image approximation (480x640)

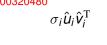




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Image approximation (480x640)







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