Chapter 6: Lecture 25

Linear Algebra, Course 124B, Fall, 2008

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Outline

All the way with $A\vec{x} = \vec{b}$:

- Applies to any $m \times n$ matrix A.
- Symmetry of A and A^{T}.

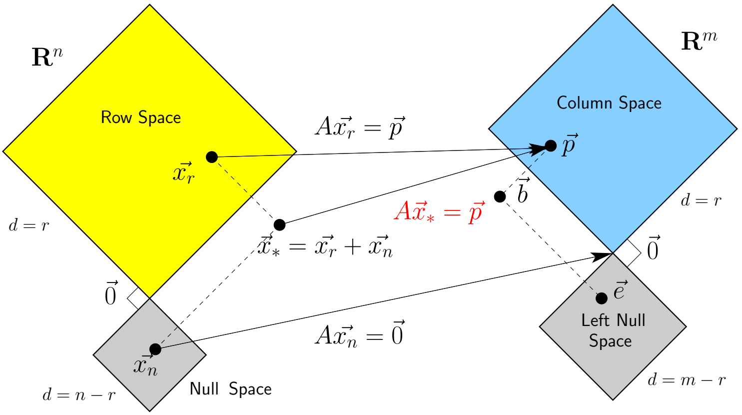
Where $\vec{x}$ lives:

- Row space $C(A^{\mathrm{T}}) \subset R^n$.
- (Right) Nullspace $N(A) \subset R^n$.
- dim $C(A^{\mathrm{T}})$ + dim $N(A)$ = $r + (n - r) = n$
- Orthogonality: $C(A^{\mathrm{T}}) \bigotimes N(A) = R^n$

Where $\vec{b}$ lives:

- Column space $C(A) \subset R^m$.
- Left Nullspace $N(A^{\mathrm{T}}) \subset R^m$.
- dim $C(A)$ + dim $N(A^{\mathrm{T}})$ = $r + (m - r) = m$
- Orthogonality: $C(A) \bigotimes N(A^{\mathrm{T}}) = R^m$

Best solution $\vec{x}_*$ when $\vec{b} = \vec{p} + \vec{e}$:



Fundamental Theorem of Linear Algebra

Now we see:

- Each of the four fundamental subspaces has a 'best' orthonormal basis
- The $\hat{v}_i$ span R^n
- We find the $\hat{v}_i$ as eigenvectors of $A^{\mathrm{T}}A$.
- The $\hat{u}_i$ span R^m
- We find the $\hat{u}_i$ as eigenvectors of AA^{T}.

Happy bases

- $\{\hat{v}_1, \dots, \hat{v}_r\}$ span Row space
- $\{\hat{v}_{r+1}, \dots, \hat{v}_n\}$ span Null space
- $\{\hat{u}_1, \dots, \hat{u}_r\}$ span Column space
- $\{\hat{u}_{r+1}, \dots, \hat{u}_m\}$ span Left Null space

Fundamental Theorem of Linear Algebra

How $A\vec{x}$ works:

- $A = U\Sigma V^{\mathrm{T}}$
- A sends each $\vec{v}_i \in C(A^{\mathrm{T}})$ to its partner $\vec{u}_i \in C(A)$ with a stretch/shrink factor $\sigma_i > 0$.
- A is diagonal with respect to these bases and has positive entries (all $\sigma_i > 0$).
- When viewed the right way, any A is a diagonal matrix Σ.

Image approximation (80x60)

Idea: use SVD to approximate images

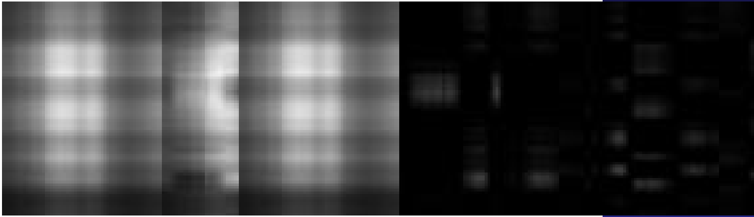
- Interpret elements of matrix A as color values of an image.
- Truncate series SVD representation of A:

$$A = U\Sigma V^{\mathrm{T}} = \sum_{i=1}^{r} \sigma_i \hat{u}_i \hat{v}_i^{\mathrm{T}}$$

- Use fact that $\sigma_1 > \sigma_2 > \ldots > \sigma_r > 0$.
- Rank $r = \min(m, n)$.
- Rank $r =$ # of pixels on shortest side.
- For color: approximate 3 matrices (RGB).

Image approximation (80x60)

$$A = \sum_{i=1}^{1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,20\,30\,40\,50\,60} \sigma_i \hat{u}_i \hat{v}_i^{\mathrm{T}}$$



Decay of sigma values: Einstein

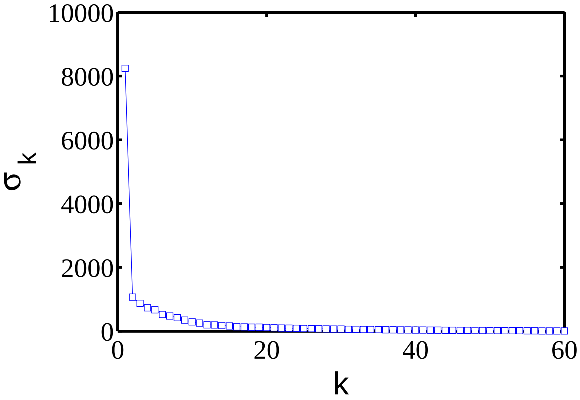


Image approximation (480x615)

12345678910203040506048O

$$A = \sum_{i=1} \sigma_i \hat{u}_i \hat{v}_i^{\mathrm{T}}$$

Image approximation (480x640)

$$A = \sum_{i=1}^{123456789102030405060100200320480} \sigma_i \hat{u}_i \hat{v}_i^{\mathrm{T}}$$

Image approximation (480x640)

$$A = \sum_{i=1}^{12345678910203040506010020032048 0} \sigma_i \hat{u}_i \hat{v}_i^{\mathrm{T}}$$