Chapter 2: Lecture 2

Linear Algebra, Course 124B, Fall, 2008

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Outline

Solving $A\vec{x} = \vec{b}$

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Reading:

- Presumed: 1.1 and 1.2
- First week: 2.1, 2.2
- Next Tuesday: 2.3

Lectures online:

- Gil Strang speaks (⊞) (18.06 at MIT, 2006).

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- We (people + computers) solve systems of linear equations by a systematic method of Elimination followed by Back substitution
- Due to our man Gauss, hence Gaussian elimination.
- Our first example:

$$\begin{array}{rcrcl} -x_1 & + & 3x_2 & = & 1 \\ 2x_1 & + & x_2 & = & 5 \end{array}$$

Gaussian elimination:

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   e.g.

$$\begin{array}{rcrcr} & & x_2 & = & 3 \\ 2x_1 & - & x_2 & = & -1 \end{array} \Rightarrow \begin{array}{rcrcr} 2x_1 & - & x_2 & = & -1 \\ & & x_2 & = & 3 \end{array}$$

Gaussian elimination:

Solve:

$$2x - 3y = 3$$
$$4x - 5y + z = 7$$
$$2x - y - 3z = 5$$

Gaussian elimination:

Summary:

Using row operations, we turned this problem:

$$A\vec{x} = \vec{b} : \begin{bmatrix} 2 & -3 & 0 \\ 4 & -5 & 1 \\ 2 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}$$

into this problem:

$$U\vec{x} = \vec{d} : \begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

and the latter is easy to solve using back substitution.

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Truth:

If at least one pivot is zero, the matrix will be singular (but the reverse is not necessarily true).

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- Eliminate entries one column at a time, moving left to right, and down each column.

$$\begin{array}{ccccccccc}
X & + & X & + & X & + & X & = & X \\
1\downarrow & + & X & + & X & + & X & = & X \\
2\downarrow & + & 4\downarrow & + & X & + & X & = & X \\
3\nearrow & + & 5\rightarrow & + & 6 & + & X & = & X
\end{array}$$

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- For example:

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$\ell_{21} = 1/2$, $\ell_{31} = -1/2$, $\ell_{41} = ?$.

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- Note: the denominator of each ℓ_{ij} multiplier is the pivot in the *j*th column.