Chapter 2: Lecture 1

Linear Algebra, Course 124B, Fall, 2008

Prof. Peter Dodds

Department of Mathematics & Statistics
University of Vermont

Outline

Basics:

- Instructor: Prof. Peter Dodds
- Lecture room and meeting times:
  111 Lafayette, Tuesday and Thursday, 2:00 pm to 3:15 pm
- Office: 203 Lord House, 16 Colchester Avenue
- E-mail: pdodds@uvm.edu
- Course website:
  `http://www.uvm.edu/~pdodds/teaching/courses/2008-08UVM-124/`
- Textbook: “Introduction to Linear Algebra” (3rd ed.) by Gilbert Strang; Wellesley-Cambridge Press.

Admin:

Paper products:

1. Outline
2. “The Fundamental Theorem of Linear Algebra” [1]
3. “Too Much Calculus” [2]

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Office hours:

- 9:00 am to 10:30 am
  Tuesday and Thursday
  Rm 203, Math Building

Grading breakdown:

1. Assignments (40%)
   - Ten one-week assignments.
   - Lowest assignment score will be dropped.
   - The last assignment cannot be dropped!
   - Each assignment will have a random bonus point question which has nothing to do with linear algebra.

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2. Midterm exams (35%)
   - Three 75 minutes tests distributed throughout the course, all of equal weighting.
3. Final exam (24%)
   - Three hours of pure happiness.
   - Tuesday, December 16th, 2008, 3:30 pm to 6:30 pm, in 111 Lafayette.

Grading breakdown:

1. Homework (0%)—Problems assigned online from the textbook. Doing these exercises will be most beneficial and will increase happiness.
2. General attendance (1%)—it is extremely desirable that students attend class, and class presence will be taken into account if a grade is borderline. Contributing to examples of linear algebra in action for the class blog will help too.

How grading works:

Questions are worth 3 points according to the following scale:

- 3 = correct or very nearly so.
- 2 = acceptable but needs some revisions.
- 1 = needs major revisions.
- 0 = way off.

Schedule:

The course will mainly cover chapters 2 through 6 of the textbook. (You should know all about Chapter 1.)

Week # (dates)	**Tuesday**	**Thursday**
1 (9/2, 9/4)	Lecture	Lecture ➤ A1
2 (9/9, 9/11)	Lecture	Lecture ➤ A2
3 (9/16, 9/18)	Lecture	Lecture ➤ A3
4 (9/23, 9/25)	*Review*	*Test 1*
5 (9/30, 10/2)	Lecture	Lecture ➤ A4
6 (10/7, 10/9)	Lecture	Lecture ➤ A5
7 (10/14, 10/16)	Lecture	Lecture ➤ A6
8 (10/21, 10/23)	*Review*	*Test 2*
9 (10/28, 10/30)	Lecture	Lecture ➤ A7
10 (11/4, 11/6)	Lecture	Lecture ➤ A8
11 (11/11, 11/13)	Lecture	Lecture ➤ A9
12 (11/18, 11/20)	*Review*	*Test 3*
13 (11/25, 11/27)	Thanksgiving	Thanksgiving
14 (12/2, 12/4)	Lecture	Lecture ➤ A10
15 (12/9, 12/11)	Lecture	*Review*

Important dates:

1. Classes run from Tuesday, September 2nd to Thursday, December 11.
2. Add/Drop, Audit, Pass/No Pass deadline—Monday, September 15.
3. Last day to withdraw—Friday, October 31.
4. Reading and exam period—Friday, December 12th to Friday, December 19th.

More stuff:

Do check your zoo account for updates regarding the course.

Academic assistance: Anyone who requires assistance in any way (as per the ACCESS program or due to athletic endeavors), please see or contact me as soon as possible.

More stuff:

Being good people:

1. In class there will be no electronic gadgetry, no cell phones, no beeping, no text messaging, etc. You really just need your brain, some paper, and a writing implement here (okay, and Matlab or similar).

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2. Second, I encourage you to email me questions, ideas, comments, etc., about the class but request that you please do so in a respectful fashion.
3. Finally, as in all UVM classes, Academic honesty will be expected and departures will be dealt with appropriately. See `http://www.uvm.edu/cses/` for guidelines.

More stuff:

Late policy: Unless in the case of an emergency (a real one) or if an absence has been predeclared and a make-up version sorted out, assignments that are not turned in on time or tests that are not attended will be given 0%.

Computing: Students are encouraged to use Matlab or something similar to check their work.

Note: for assignment problems, written details of calculations will be required.

Grading:

A+	97–100	B+	87–89	C+	77–79	D+	67–69
A	93–96	B	83–86	C	73–76	D	63–66
A-	90–92	B-	80–82	C-	70–72	D-	60–62

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a body of mathematics
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Even more:

If real data is continuous, we almost always discretize it (0's and 1's)

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Linear Algebra is used in many fields to solve problems:

- Engineering
- Computer Science (Google's Pagerank)
- Physics
- Economics
- Biology
- Ecology
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Linear Algebra is as important as calculus.

Matrices as gadgets:

A transforms $\vec{x}$ into $\vec{x}'$ through multiplication

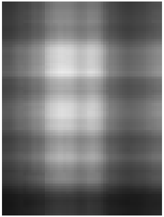
$$\vec{x}' = A\vec{x}$$

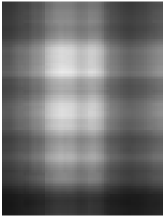
Can use matrices to:

- Grow vectors
- Shrink vectors
- Rotate vectors
- Flip vectors
- Do all these things to different directions

Image approximation (80x60)

$$A = \sum_{i=1}^{1} \sigma_i \hat{u}_i \hat{v}_i^{\mathrm{T}}$$





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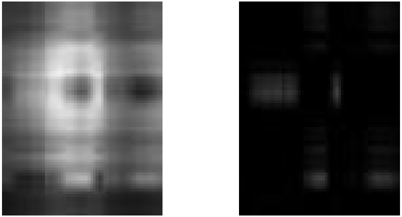

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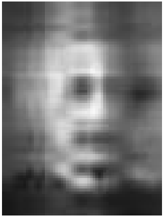

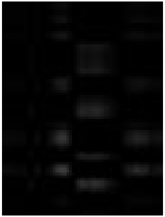

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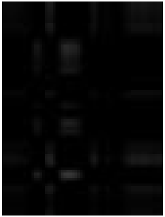

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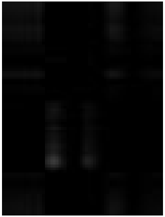

Image approximation (80x60)

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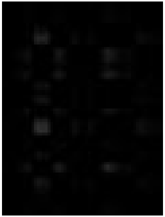

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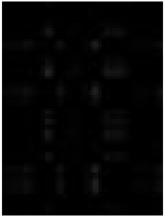
$$A = \sum_{i=1}^{7} \sigma_i \hat{u}_i \hat{v}_i^{\mathrm{T}}$$



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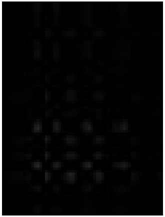

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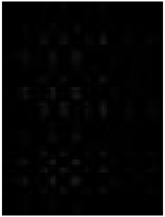
$$A = \sum_{i=1}^{9} \sigma_i \hat{u}_i \hat{v}_i^{\mathrm{T}}$$



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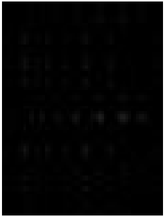
$$A = \sum_{i=1}^{\color{red}{10}} \sigma_i \hat{u}_i \hat{v}_i^{\mathrm{T}}$$



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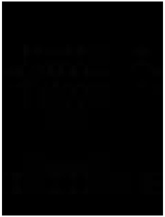

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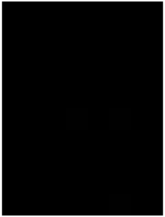

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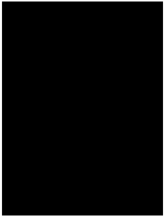

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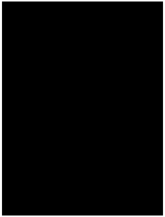

Image approximation (80x60)

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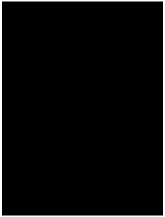

Three key problems of Linear Algebra

1. Given a matrix A and a vector $\vec{b}$, find $\vec{x}$ such that

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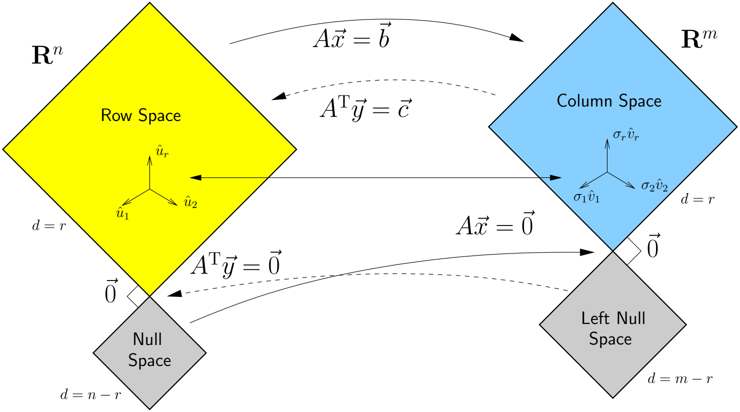
- Our focus will be largely on #1, partly on #2.

Major course objective:

To deeply understand the equation $A\vec{x} = \vec{b}$, the Fundamental Theorem of Linear Algebra, and the following picture:



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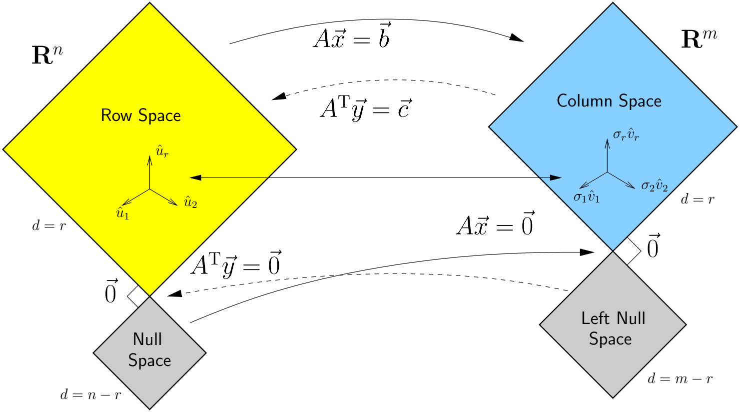

What is going on here? We have 26 lectures to find out...

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How can we disentangle an orchestra's sound?

What about pictures, waves, signals, ...?

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If we can represent reality as a superposition (or combination) of simple elements, we can do many things:

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- See how we can alter information
- Find a system's simplest representation
- Find a system's most important elements
- See how to adjust a system in a principled defined way

Three ways to understand $A\vec{x} = \vec{b}$:

- Way 1: The Row Picture
- Way 2: The Column Picture
- Way 3: The Matrix Picture

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Three possible kinds of solution:

1. Lines intersect at one point —One, unique solution
2. Lines are parallel and disjoint —No solutions
3. Lines are the same —Infinitely many solutions

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- Column vectors are 'building blocks'
- Key idea: try to 'reach' $\vec{b}$ by combining multiples of column vectors $\vec{a}_1$ and $\vec{a}_2$.

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Assuming neither $\vec{a}_1$ or $\vec{a}_1$ are $\vec{0}$.

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- Method of approximation—very important!
- We may not have the right building blocks but we can do our best.

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- A transforms $\vec{x}$ into $\vec{b}$.
- In general, A does two things to $\vec{x}$:
  1. Rotation
  2. Dilation (stretching/contraction)

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- $A = LU$, $A = QR$, $A = U\Sigma V^{\mathrm{T}}$, $A = \sum_i \lambda_i \vec{v}\vec{v}^{\mathrm{T}}$, ...

The truth about mathematics

The Colbert Report on Math (February 7, 2006)

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