Models of Complex Networks

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Principles of Complex Systems, Vols. 1, 2, & 3D CSYS/MATH 6701, 6713, & a pretend number, 2023–2024| @pocsvox

Prof. Peter Sheridan Dodds | @peterdodds

Computational Story Lab | Vermont Complex Systems Center
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Outline

Models

Some important models:

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1. Generalized random networks

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4. Statistical generative models (p^*)

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3. Small-world networks
4. Statistical generative models (p^*)
5. Generalized affiliation networks

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- Interesting, applicable, rich mathematically.
- Much fun to be had with these guys...

Models

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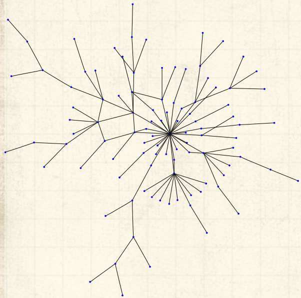

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$\langle k \rangle$ = 1.8
$N = 150$

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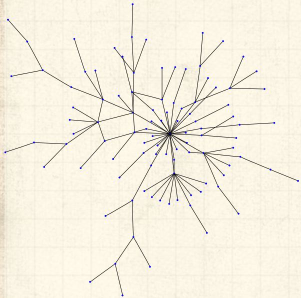

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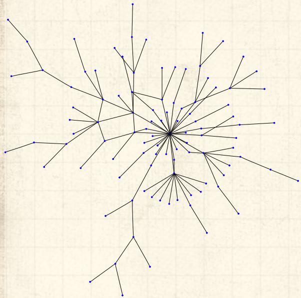

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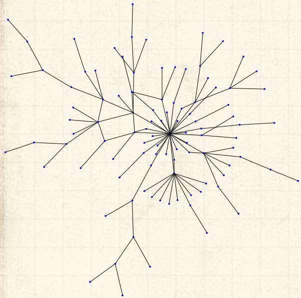

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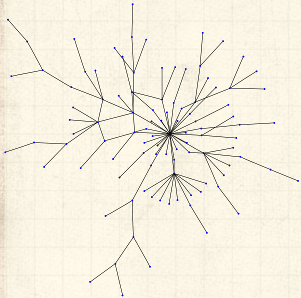

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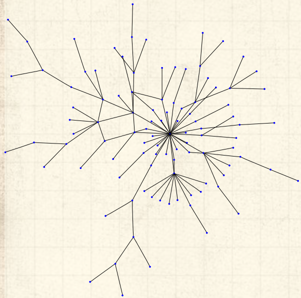

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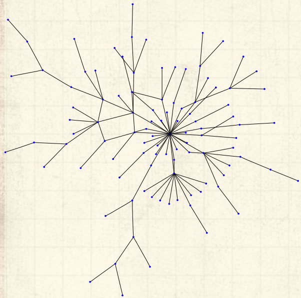

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- Trickiness: other models generate skewed degree distributions...

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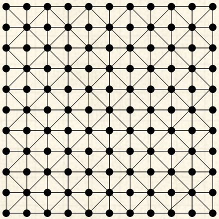

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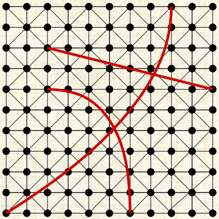

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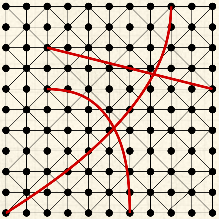

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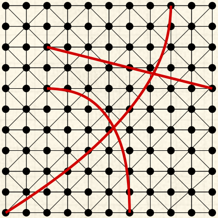

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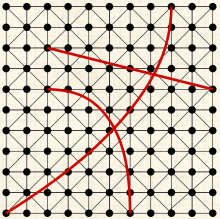
- Facilitates synchronization [8]



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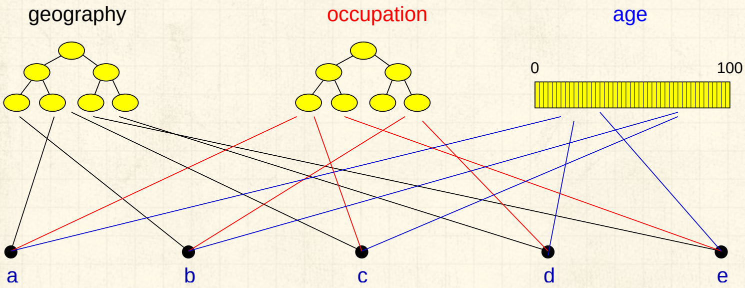
- Use statistical methods to estimate 'best' values of parameters.
- Drawback: parameters are not real, measurable quantities.
- Non-mechanistic and blackboxish.
- c.f., temperature in statistical mechanics.

Models

5. Generalized affiliation networks



- Blau & Schwartz [3], Simmel [15], Breiger [4], Watts *et al.* [17]

Outline

Pure, abstract random networks:

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- Known as Erdős-Rényi random networks
- Key structural feature of random networks is that they locally look like branching networks
- (No small cycles and zero clustering).

Random networks: examples

Next slides:

Example realizations of random networks

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- $N = 500$

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- $N = 500$
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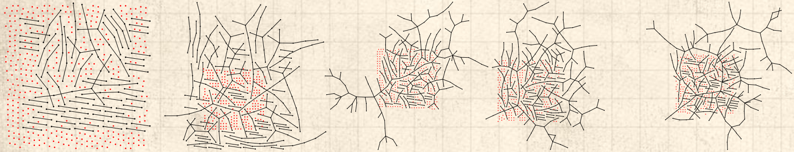
Example realizations of random networks

- $N = 500$
- Vary m, the number of edges from 100 to 1000.
- Average degree $\langle k \rangle$ runs from 0.4 to 4.
- Look at full network plus the largest component.

Random networks: examples for N=500



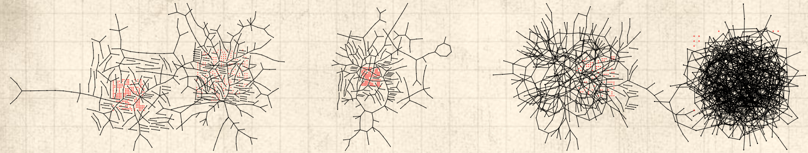
m = 100 $\langle k \rangle$ = 0.4

m = 200 $\langle k \rangle$ = 0.8

m = 230 $\langle k \rangle$ = 0.92

m = 240 $\langle k \rangle$ = 0.96

m = 250 $\langle k \rangle$ = 1



m = 260 $\langle k \rangle$ = 1.04

m = 280 $\langle k \rangle$ = 1.12

m = 300 $\langle k \rangle$ = 1.2

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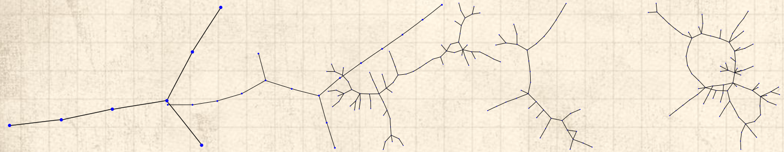
Random networks: largest components



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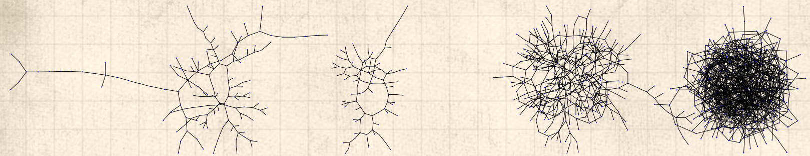

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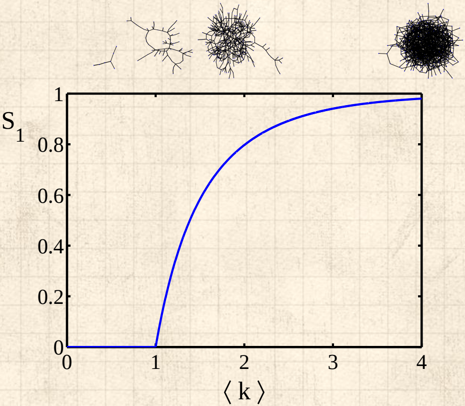
Giant component:



- S_1 = fraction of nodes in largest component.
- Old school phase transition.
- Key idea in modeling contagion.

Properties

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But but:

- Randomness is out there, just not to the degree of a completely random network.

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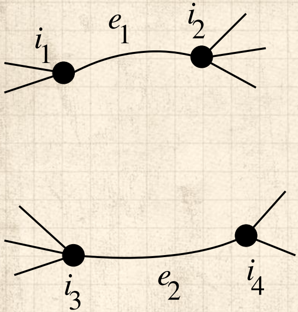
  2. Examine mechanisms that lead to networks with certain degree distributions.

General random rewiring algorithm



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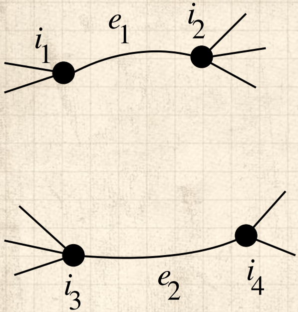

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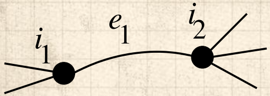

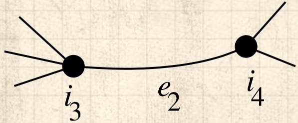

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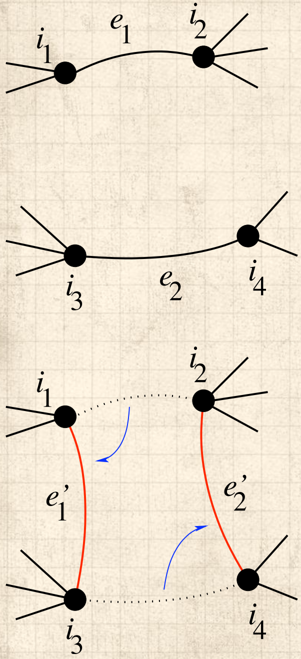

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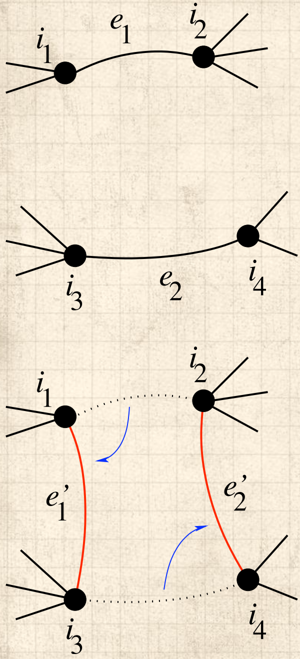

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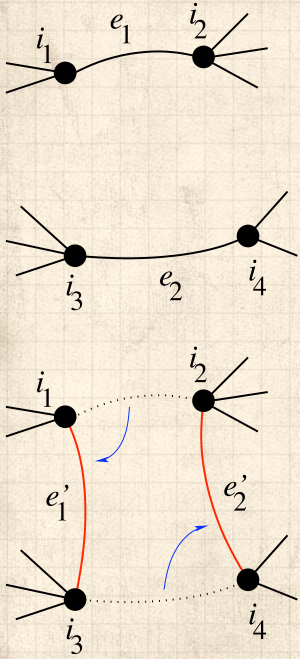

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- Same as finding on/off/on/off 4-cycles. and rotating them.

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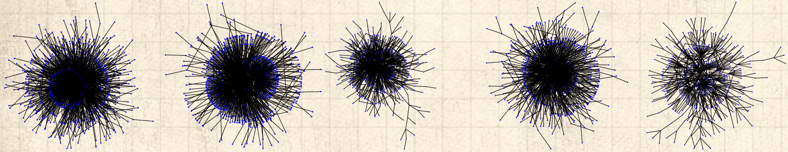
- Apart from degree distribution, wiring is random.

Random networks: largest components



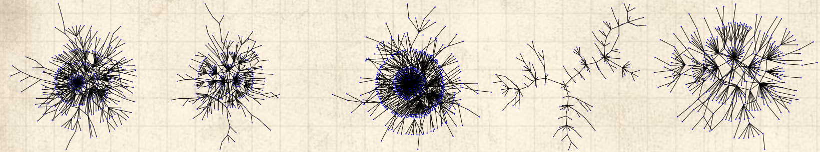
γ = 2.1
$\langle k \rangle$ = 3.448

γ = 2.19
$\langle k \rangle$ = 2.986

γ = 2.28
$\langle k \rangle$ = 2.306

γ = 2.37
$\langle k \rangle$ = 2.504

γ = 2.46
$\langle k \rangle$ = 1.856



γ = 2.55
$\langle k \rangle$ = 1.712

γ = 2.64
$\langle k \rangle$ = 1.6

γ = 2.73
$\langle k \rangle$ = 1.862

γ = 2.82
$\langle k \rangle$ = 1.386

γ = 2.91
$\langle k \rangle$ = 1.49

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- Natural question: what's the expected number of other friends that one friend has?

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  R_k = probability that a friend of a random node has k other friends.
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- True for all random networks, independent of degree distribution.

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- We'll see this again for contagion models...

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  3. Your friends have more friends than you...

Size distributions

The sizes of many systems' elements appear to obey an
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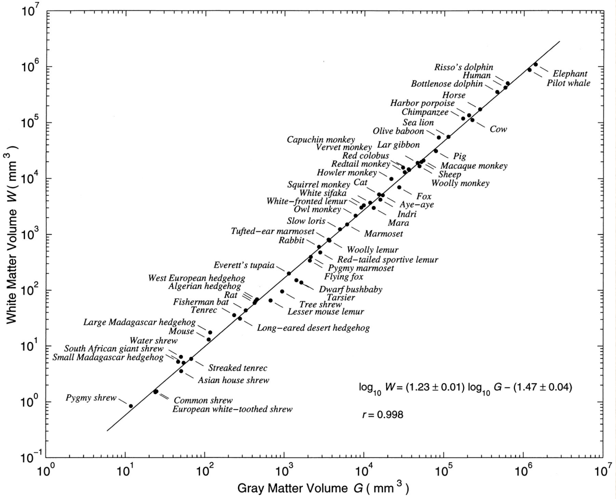
- Negative linear relationship in log-log space:

$$\log P(x) = \log c - \gamma \log x$$

A beautiful, heart-warming example:



from Zhang & Sejnowski, PNAS (2000) [19]

$\alpha \simeq 1.23$

gray matter: 'computing elements'

white matter: 'wiring'

Size distributions

Power law size distributions are sometimes called Pareto distributions after Italian scholar Vilfredo Pareto.

- Pareto noted wealth in Italy was distributed unevenly (80–20 rule).
- Term used especially by economists

Size distributions

Examples:

- Earthquake magnitude (Gutenberg Richter law): $P(M) \propto M^{-3}$
- Number of war deaths: $P(d) \propto d^{-1.8}$ [14]
- Sizes of forest fires
- Sizes of cities: $P(n) \propto n^{-2.1}$
- Number of links to and from websites

Size distributions

Examples:

- Number of citations to papers: $P(k) \propto k^{-3}$.
- Individual wealth (maybe): $P(W) \propto W^{-2}$.
- Distributions of tree trunk diameters: $P(d) \propto d^{-2}$.
- The gravitational force at a random point in the universe: $P(F) \propto F^{-5/2}$.
- Diameter of moon craters: $P(d) \propto d^{-3}$.
- Word frequency: e.g., $P(k) \propto k^{-2.2}$ (variable)

Note: Exponents range in error;
see M.E.J. Newman arxiv.org/cond-mat/0412004v3

Outline

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- 1999: Barabasi and Albert [2]:
  The World Wide Web, networks-at-large.

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- Please note: authors do not do this...

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- See γ is governed by rate of new flavor creation, ρ.

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And just to rub it in...

Merton's son, Robert C. Merton, won the Nobel Prize for Economics in 1997.

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- Scale-free networks = food on the table for physicists

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- Please note: not every network is a scale-free network...

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- Usually talking about networks whose links are abstract, relational, informational, ...(non-physical)

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- Much arguing about whether or networks are ‘scale-free’ or not...

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- We know they do for Simon's model...

Real data (eek!)

From Barabási and Albert's original paper [2]:

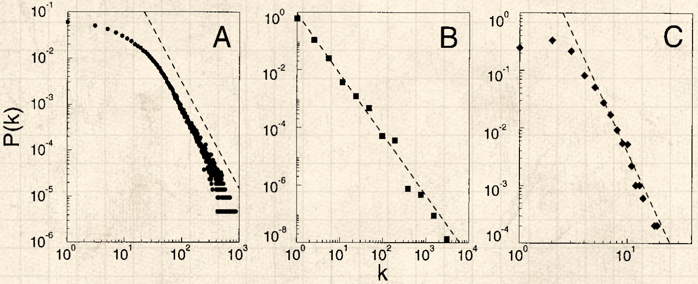

Fig. 1. The distribution function of connectivities for various large networks. (**A**) Actor collaboration graph with $N = 212{,}250$ vertices and average connectivity $\langle k \rangle = 28.78$. (**B**) WWW, $N = 325{,}729$, $\langle k \rangle = 5.46$ (6). (**C**) Power grid data, $N = 4941$, $\langle k \rangle = 2.67$. The dashed lines have slopes (A) $\gamma_{\text{actor}} = 2.3$, (B) $\gamma_{\text{www}} = 2.1$ and (C) $\gamma_{\text{power}} = 4$.

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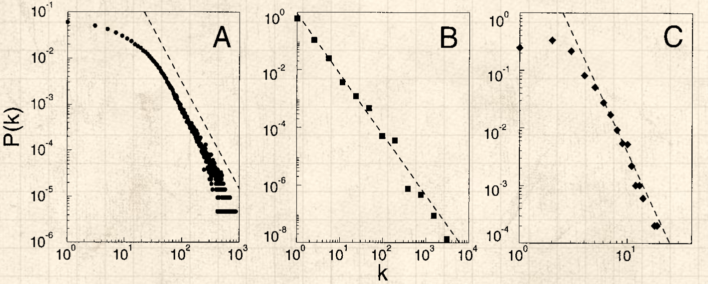

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- (Plot C is on the bogus side of things...)

Outline

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- RK's approach is to use rate equations.

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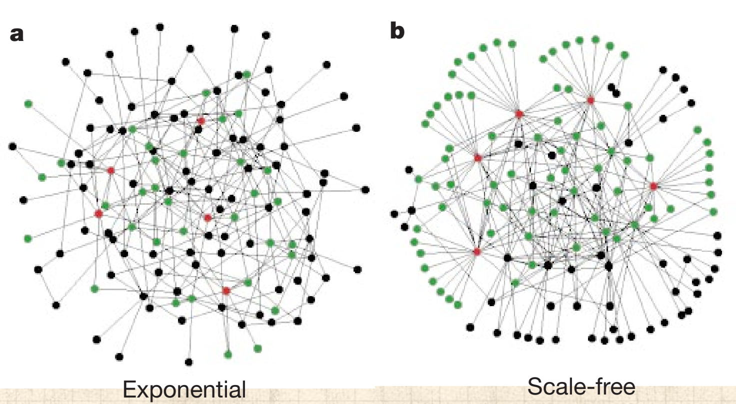
- For $\nu > 2$, all but a finite # of nodes connect to one node.

Outline

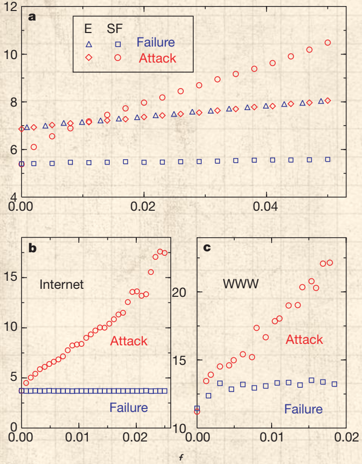
Robustness

- Standard random networks (Erdős-Rényi) versus Scale-free networks



from Albert et al., 2000 "Error and attack tolerance of complex networks" [1]

Robustness



from Albert et al., 2000

- Plots of network diameter as a function of fraction of nodes removed
- Erdős-Rényi versus scale-free networks
- blue symbols = random removal
- red symbols = targeted removal (most connected first)

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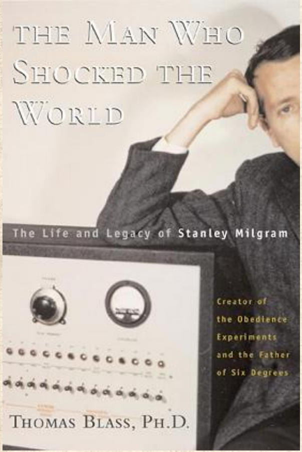
- Need to explore cost of various targeting schemes.

Outline

Milgram's social search experiment (1960s)



http://www.stanleymilgram.com

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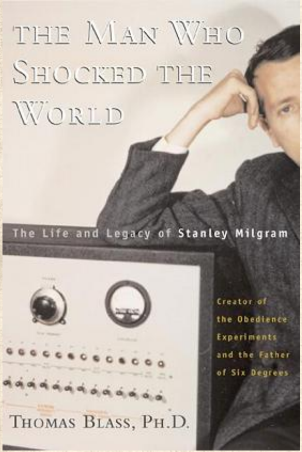

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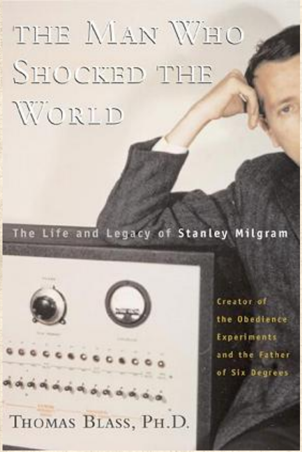

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Popular terms:

- The Small World Phenomenon;
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18 targets in 13 countries including

- a professor at an Ivy League university,
- an archival inspector in Estonia,
- a technology consultant in India,
- a policeman in Australia,
- a potter in New Zealand,
- a veterinarian in the Norwegian army.

PoCS
Principles of Complex Systems
@pocsvox
What's the Story?

Social search—the Columbia experiment

The world is smaller:

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- c.f. Milgram (zero attrition): $L_* \simeq 9$

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Previous work—short paths

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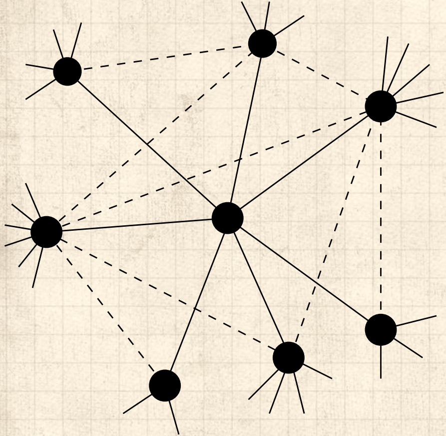

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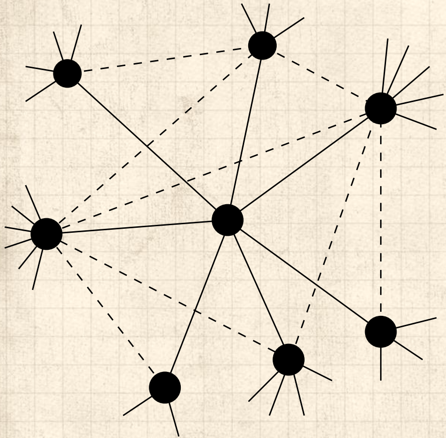

- Need "clustering" (your friends are likely to know each other):
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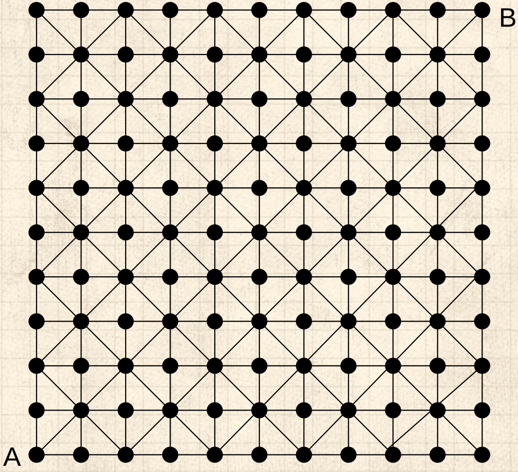
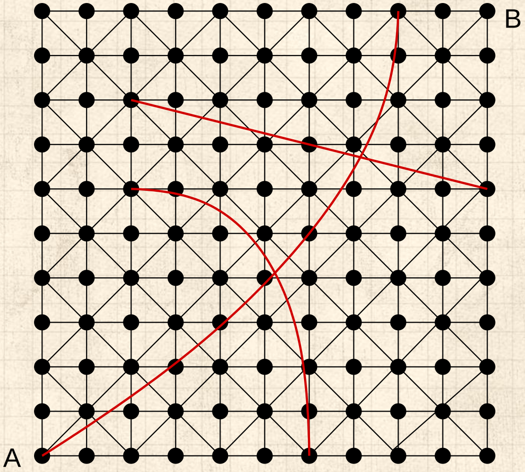
Non-randomness gives clustering



$d_{AB} = 10 \rightarrow$ too many long paths.

Randomness + regularity



$d_{AB} = 10$ without random paths
$d_{AB} = 3$ with random paths

$\langle d \rangle$ decreases overall

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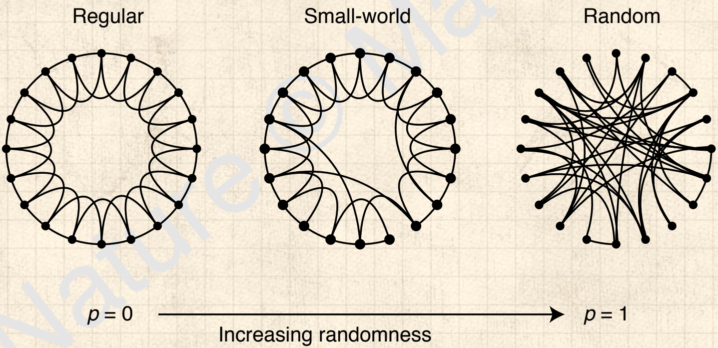
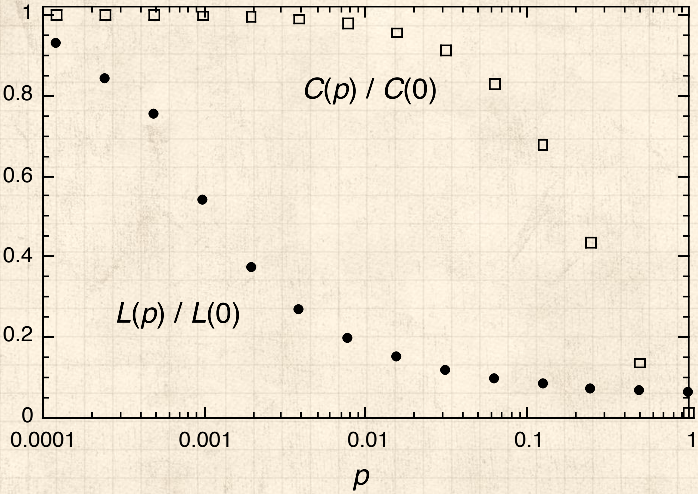
Very weak requirements:

- local regularity + random short cuts

Toy model



The structural small-world property



The structural small-world property

Table 1 Empirical examples of small-world networks

	L_{actual}	L_{random}	C_{actual}	C_{random}
Film actors	3.65	2.99	0.79	0.00027
Power grid	18.7	12.4	0.080	0.005
C. elegans	2.65	2.25	0.28	0.05

Characteristic path length L and clustering coefficient C for three real networks, compared to random graphs with the same number of vertices (n) and average number of edges per vertex (k). (Actors: $n = 225{,}226$, $k = 61$. Power grid: $n = 4{,}941$, $k = 2.67$. *C. elegans*: $n = 282$, $k = 14$.) The graphs are defined as follows. Two actors are joined by an edge if they have acted in a film together. We restrict attention to the giant connected component[16] of this graph, which includes ~90% of all actors listed in the Internet Movie Database (available at http://us.imdb.com), as of April 1997. For the power grid, vertices represent generators, transformers and substations, and edges represent high-voltage transmission lines between them. For *C. elegans*, an edge joins two neurons if they are connected by either a synapse or a gap junction. We treat all edges as undirected and unweighted, and all vertices as identical, recognizing that these are crude approximations. All three networks show the small-world phenomenon: $L \gtrsim L_{\text{random}}$ but $C \gg C_{\text{random}}$.

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  "Navigation in a small world."
- Only certain networks are navigable
- So what's special about social networks?

The model

One approach: incorporate identity.

(See "Identity and Search in Social Networks." Science, 2002, Watts, Dodds, and Newman [17])

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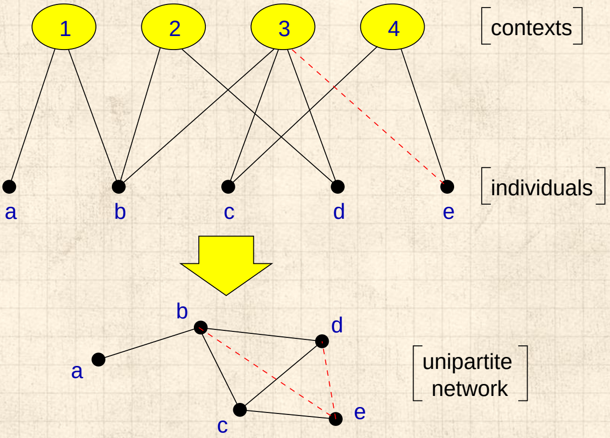
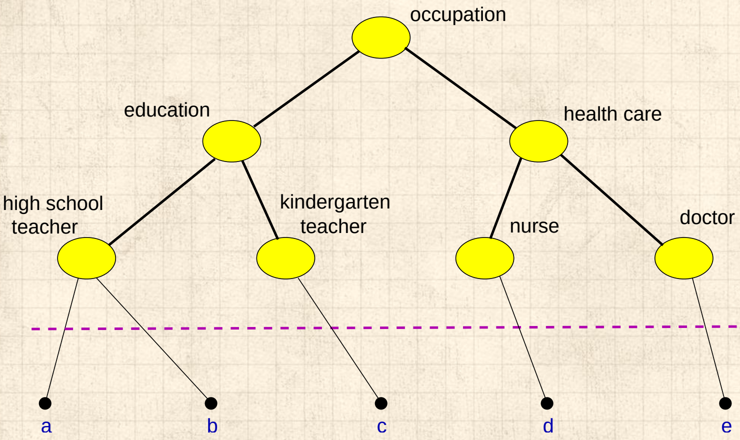
Attributes ⇔ Contexts ⇔ Interactions ⇔ Networks.

Social distance—Bipartite affiliation networks

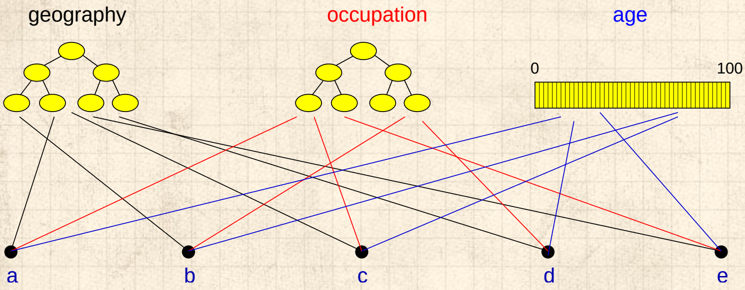


Bipartite affiliation networks: boards and directors, movies and actors.

Social distance as a function of identity



Homophily



(Blau & Schwartz, Simmel, Breiger)

- Networks built with 'birds of a feather...' are searchable.
- Attributes ⇔ Contexts ⇔ Interactions ⇔ Networks.

Social Search—Real world uses

- Tagging: e.g., Flickr induces a network between photos
- Search in organizations for solutions to problems
- Peer-to-peer networks
- Synchronization in networked systems
- Motivation for search matters...

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