# Measures of centrality

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# Outline

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- Degree centrality
- Closeness centrality
- Betweenness
- Eigenvalue centrality
- Hubs and Authorities

Nutshell

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The Manifesto
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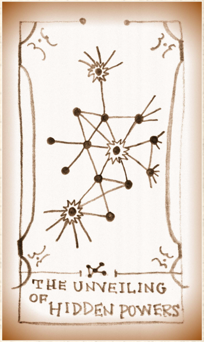

The Unveiling of Hidden Powers

# How big is my node?





- Basic question: how 'important' are specific nodes and edges in a network?
- An important node or edge might:
  1. handle a relatively large amount of the network's traffic (e.g., cars, information);
  2. bridge two or more distinct groups (e.g., liason, interpreter);
  3. be a source of important ideas, knowledge, or judgments (e.g., supreme court decisions, an employee who 'knows where everything is').
- So how do we quantify such a slippery concept as importance?
- We generate ad hoc, reasonable measures, and examine their utility ...

# Centrality





- One possible reflection of importance is centrality.
- Presumption is that nodes or edges that are (in some sense) in the middle of a network are important for the network's function.
- Idea of centrality comes from social networks literature [7].
- Many flavors of centrality ...
  1. Many are topological and quasi-dynamical;
  2. Some are based on dynamics (e.g., traffic).
- We will define and examine a few ...
- (Later: see centrality useful in identifying communities in networks.)

# Centrality





## Degree centrality

- Naively estimate importance by node degree. [7]
- Doh: assumes linearity
  (If node $i$ has twice as many friends as node $j$, it's twice as important.)
- Doh: doesn't take in any non-local information.

# Closeness centrality





- Idea: Nodes are more central if they can reach other nodes 'easily.'
- Measure average shortest path from a node to all other nodes.
- Define Closeness Centrality for node $i$ as

$$\frac{N-1}{\sum_{j,j\neq i}(\text{shortest distance from } i \text{ to } j).}$$

- Range is 0 (no friends) to 1 (single hub).
- Unclear what the exact values of this measure tells us because of its ad-hocness.
- General problem with simple centrality measures: what do they exactly mean?
- Perhaps, at least, we obtain an ordering of nodes in terms of 'importance.'

# Betweenness centrality

- Betweenness centrality is based on coherence of shortest paths in a network.
- Idea: If the quickest way between any two nodes on a network disproportionately involves certain nodes, then they are 'important' in terms of global cohesion.
- For each node $i$, count how many shortest paths pass through $i$.
- In the case of ties, divide counts between paths.
- Call frequency of shortest paths passing through node $i$ the betweenness of $i$, $B_i$.
- Note: Exclude shortest paths between $i$ and other nodes.
- Note: works for weighted and unweighted networks.

- Consider a network with $N$ nodes and $m$ edges (possibly weighted).
- Computational goal: Find $\binom{N}{2}$ shortest paths between all pairs of nodes.
- Traditionally use Floyd-Warshall algorithm.
- Computation time grows as $O(N^3)$.
- See also:
  1. Dijkstra's algorithm for finding shortest path between two specific nodes,
  2. and Johnson's algorithm which outperforms Floyd-Warshall for sparse networks: $O(mN + N^2 \log N)$.
- Newman (2001) [4, 5] and Brandes (2001) [1] independently derive equally fast algorithms that also compute betweenness.
- Computation times grow as:
  1. $O(mN)$ for unweighted graphs;
  2. and $O(mN + N^2 \log N)$ for weighted graphs.

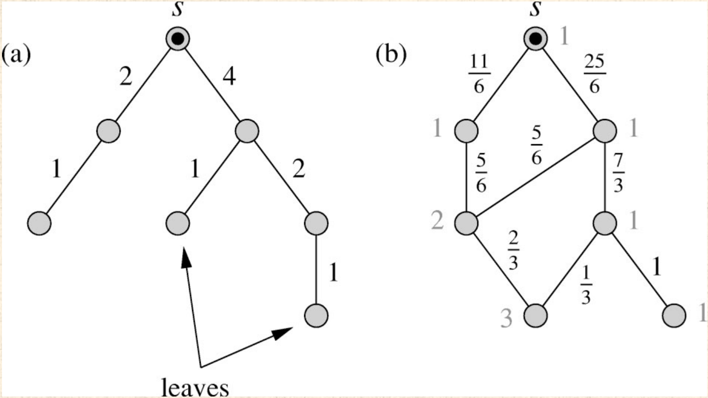
# Shortest path between node $i$ and all others:

- Consider unweighted networks.
- Use breadth-first search:
    1. Start at node $i$, giving it a distance $d = 0$ from itself.
    2. Create a list of all of $i$'s neighbors and label them being at a distance $d = 1$.
    3. Go through list of most recently visited nodes and find all of their neighbors.
    4. Exclude any nodes already assigned a distance.
    5. Increment distance $d$ by 1.
    6. Label newly reached nodes as being at distance $d$.
    7. Repeat steps 3 through 6 until all nodes are visited.
- Record which nodes link to which nodes moving out from $i$ (former are 'predecessors' with respect to $i$'s shortest path structure).
- Runs in $O(m)$ time and gives $N - 1$ shortest paths.
- Find all shortest paths in $O(mN)$ time

# Newman's Betweenness algorithm: [4]







# Newman's Betweenness algorithm: [4]

1. Set all nodes to have a value $c_{ij} = 0$, $j = 1, ...$ ($c$ for count).
2. Select one node $i$ and find shortest paths to all other $N - 1$ nodes using breadth-first search.
3. Record # equal shortest paths reaching each node.
4. Move through nodes according to their distance from $i$, starting with the furthest.
5. Travel back towards $i$ from each starting node $j$, along shortest path(s), adding 1 to every value of $c_{i\ell}$ at each node $\ell$ along the way.
6. Whenever more than one possibility exists, apportion according to total number of short paths coming through predecessors.
7. Exclude starting node $j$ and $i$ from increment.
8. Repeat steps 2–8 for every node $i$ and obtain betweenness as $B_j = \sum_{i=1}^{N} c_{ij}$.

# Newman's Betweenness algorithm: [4]

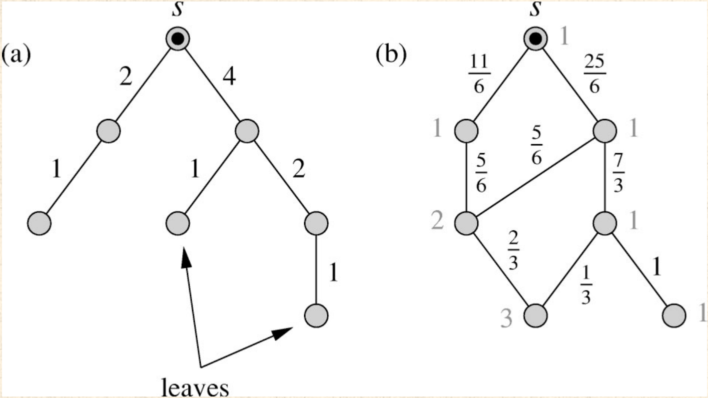




- For a pure tree network, $c_{ij}$ is the number of nodes beyond $j$ from $i$'s vantage point.
- Same algorithm for computing drainage area in river networks (with 1 added across the board).
- For edge betweenness, use exact same algorithm but now
  1. $j$ indexes edges,
  2. and we add one to each edge as we traverse it.
- For both algorithms, computation time grows as

$$O(mN).$$

# Newman's Betweenness algorithm: [4]



# Important nodes have important friends:

- Define $x_i$ as the ‘importance’ of node $i$.
- Idea: $x_i$ depends (somehow) on $x_j$
  if $j$ is a neighbor of $i$.
- Recursive: importance is transmitted through a network.
- Simplest possibility is a linear combination:

$$x_i \propto \sum_j a_{ji} x_j$$

- Assume further that constant of proportionality, $c$, is independent of $i$.
- Above gives $\vec{x} = c\mathbf{A}^\mathsf{T}\vec{x}$ or $\boxed{\mathbf{A}^\mathsf{T}\vec{x} = c^{-1}\vec{x} = \lambda\vec{x}}$.
- Eigenvalue equation based on adjacency matrix ...
- Note: Lots of despair over size of the largest eigenvalue. [7] Lose sight of original assumption’s non-physicality.

# Important nodes have important friends:

- So: solve $\mathbf{A}^{\mathsf{T}}\vec{x} = \lambda\vec{x}$.
- But which eigenvalue and eigenvector?
- We, the people, would like:
  1. A unique solution. ✔
  2. $\lambda$ to be real. ✔
  3. Entries of $\vec{x}$ to be real. ✔
  4. Entries of $\vec{x}$ to be non-negative. ✔
  5. $\lambda$ to actually mean something ... (maybe too much)
  6. Values of $x_i$ to mean something (what does an observation that $x_3 = 5x_7$ mean?) (maybe only ordering is informative ...) (maybe too much)
  7. $\lambda$ to equal 1 would be nice ... (maybe too much)
  8. Ordering of $\vec{x}$ entries to be robust to reasonable modifications of linear assumption (maybe too much)
- We rummage around in bag of tricks and pull out the Perron-Frobenius theorem ...

## Perron-Frobenius theorem: If an $N \times N$ matrix $A$ has non-negative entries then:

1. $A$ has a real eigenvalue $\lambda_1 \geq |\lambda_i|$ for $i = 2, \ldots, N$.
2. $\lambda_1$ corresponds to left and right 1-d eigenspaces for which we can choose a basis vector that has non-negative entries.
3. The dominant real eigenvalue $\lambda_1$ is bounded by the minimum and maximum row sums of $A$:

$$\min_i \sum_{j=1}^{N} a_{ij} \leq \lambda_1 \leq \max_i \sum_{j=1}^{N} a_{ij}$$

4. All other eigenvectors have one or more negative entries.
5. The matrix $A$ can make toast.
6. Note: Proof is relatively short for symmetric matrices that are strictly positive [6] and just non-negative [3].

# Other Perron-Frobenius aspects:

- Assuming our network is irreducible, meaning there is only one component, is reasonable: just consider one component at a time if more than one exists.
- Irreducibility means largest eigenvalue's eigenvector has strictly non-negative entries.
- Analogous to notion of ergodicity: every state is reachable.
- (Another term: Primitive graphs and matrices.)

# Hubs and Authorities





- Generalize eigenvalue centrality to allow nodes to have two attributes:
  1. Authority: how much knowledge, information, etc., held by a node on a topic.
  2. Hubness (or Hubosity or Hubbishness or Hubtasticness): how well a node 'knows' where to find information on a given topic.
- Original work due to the legendary Jon Kleinberg. [2]
- Best hubs point to best authorities.
- Recursive: Hubs authoritatively link to hubs, authorities hubbishly link to other authorities.
- More: look for dense links between sets of 'good' hubs pointing to sets of 'good' authorities.
- Known as the HITS algorithm (Hyperlink-Induced Topics Search).

# Hubs and Authorities

- Give each node two scores:
  1. $x_i$ = authority score for node $i$
  2. $y_i$ = hubtasticness score for node $i$
- As for eigenvector centrality, we connect the scores of neighboring nodes.
- New story I: a good authority is linked to by good hubs.
- Means $x_i$ should increase as $\sum_{j=1}^{N} a_{ji} y_j$ increases.
- Note: indices are $ji$ meaning $j$ has a directed link to $i$.
- New story II: good hubs point to good authorities.
- Means $y_i$ should increase as $\sum_{j=1}^{N} a_{ij} x_j$ increases.
- Linearity assumption:

$$\vec{x} \propto A^T \vec{y} \text{ and } \vec{y} \propto A \vec{x}$$

# Hubs and Authorities





- So let's say we have

$$\vec{x} = c_1 A^T \vec{y} \text{ and } \vec{y} = c_2 A \vec{x}$$

where $c_1$ and $c_2$ must be positive.

- Above equations combine to give

$$\vec{x} = c_1 A^T c_2 A \vec{x} = \lambda A^T A \vec{x}.$$

where $\lambda = c_1 c_2 > 0$.

- It's all good: we have the heart of singular value decomposition before us ...

# We can do this:

- $A^T A$ is symmetric.
- $A^T A$ is semi-positive definite so its eigenvalues are all $\geq 0$.
- $A^T A$'s eigenvalues are the square of $A$'s singular values.
- $A^T A$'s eigenvectors form a joyful orthogonal basis.
- Perron-Frobenius tells us that only the dominant eigenvalue's eigenvector can be chosen to have non-negative entries.
- So: linear assumption leads to a solvable system.
- What would be very good: find networks where we have independent measures of node 'importance' and see how importance is actually distributed.

## Nutshell:

- Measuring centrality is well motivated if hard to carry out well.
- We've only looked at a few major ones.
- Methods are often taken to be more sophisticated than they really are.
- Centrality can be used pragmatically to perform diagnostics on networks (see structure detection).
- Focus on nodes rather than groups or modules is a homo narrativus constraint.
- Possible that better approaches will be developed.

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